ATTRACTING BASINS OF VOLUME PRESERVING AUTOMORPHISMS OF $\mathbb{C}^k$

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Abstract. We study topological properties of attracting sets for automorphisms of $\mathbb{C}^k$. Our main result is that a generic volume preserving automorphism has a hyperbolic fixed point with a dense stable manifold. We prove the same result for volume preserving maps tangent to the identity. On the other hand, we show that an attracting set can only contain a neighborhood of the fixed point if it is an attracting fixed point. We will see that the latter does not hold in the non-autonomous setting.

1. Introduction

Let f be an automorphism of $\mathbb{C}^k$ with a fixed point at the origin. Even if the origin is not an attracting fixed point, there can still be points whose orbits converge to the origin. In this paper we will study how large such an attracting set can be. We will make this more precise later.

The behavior of an attracting set varies greatly depending on the eigenvalues of $\mathrm{d}f(0)$. If all eigenvalues have modulus strictly smaller than 1 then we say that f has an *attracting* fixed point. This is the easiest situation, the attracting set must contain a neighborhood of the origin and is biholomorphic to $\mathbb{C}^k$ [13]. The situation is similar when all eigenvalues have modulus strictly larger than 1, one just considers the inverse mapping.

The fixed point is called *hyperbolic* if no eigenvalues have modulus 1, and there are eigenvalues of modulus greater than 1 as well as less than 1. In this case the attracting set is biholomorphic to $\mathbb{C}^m$, where m is the number of eigenvalues of modulus less than 1 (this follows from [13]).

The complex structure of attracting sets has also been studied in the *semi-attracting* case (eigenvalues of modulus smaller than and equal to 1) , and for automorphisms *tangent to the identity* (where $\mathrm{d}f(0) = \mathrm{Id}$). In both cases the attracting set can also be biholomorphically equivalent to (possibly lower dimensional) complex Euclidean space, see for example [14] for the semi-attracting case and [15], [8] and [7] for automorphisms tangent to the identity.

In this article we do not study the complex structure of attracting sets but instead we look at topological properties. Suppose an automorphism has a fixed point that is not attracting but does have a non-trivial attracting set Ω. We are interested in three related questions:
(a) Can Ω be dense?
(b) Can Ω have interior points?
(c) Can Ω contain a neighborhood of the origin?

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Our main result is an affirmative answer to Question (a). More precisely, we will show the following

Theorem 1. *There is a dense G_δ-set $\mathcal{V}$ of volume preserving automorphisms of $\mathbb{C}^k$ that have a hyperbolic fixed point whose stable manifold is dense in $\mathbb{C}^k$.*

Here $\mathcal{V}$ is a dense G_δ-subset of the set of volume preserving automorphisms of $\mathbb{C}^k$, equipped with the compact open topology.

We will then focus on volume preserving automorphisms that are tangent to the identity. We will prove the existence of dense attracting sets for these maps along the same lines as for a hyperbolic fixed point.

The answer to Question (b) is obvious, since it is possible to have an attracting set that is biholomorphic to $\mathbb{C}^k$, where k is the dimension of the ambient space [14], [15], [7]. However, we will easily see that the attracting set of a *volume preserving* automorphism cannot have interior points.

We will also show that the answer to Question (c) is negative, if the attracting set contains a neighborhood of the origin then the fixed point must be attracting. We note that this result depends on the holomorphicity of the mapping, as well as on the ambient space $\mathbb{C}^k$.

Finally, we will see that in the non-autonomous setting the basin can be all of $\mathbb{C}^k$, even if all the mappings are tangent to the identity.

In the next section we will set notation and answer question (c). We will prove our main result in Section 3, and show the analogous statement for maps tangent to the identity in Section 4. In the last section we will treat the non-autonomous setting.

2. Fixed Point in the Interior

We denote by $\mathrm{Aut}(\mathbb{C}^k)$ the set of holomorphic automorphisms of $\mathbb{C}^k$ and by $\mathrm{Aut}_1(\mathbb{C}^k)$ the set of volume preserving automorphisms of $\mathbb{C}^k$, both equipped with the compact-open topology.

We let $\|\cdot\|$ denote the Euclidean norm on $\mathbb{C}^k$ and for $r > 0$ we write $B_r(z) \subset \mathbb{C}^k$ for the ball of radius r centered at z. When $z = 0$ we will just write B_r.

For $f \in \mathrm{Aut}(\mathbb{C}^k)$ with a fixed point p we will study the *attracting set*

$$\Omega = \{z \in \mathbb{C}^k \mid f^n(z) \to p, \text{ as } n \to \infty\}$$

When p is an attracting fixed point this attracting set is generally called the attracting basin, and when p is a hyperbolic fixed point it is called the stable manifold.

The automorphisms of $\mathbb{C}^k$ constructed by Ueda, Hakim and Weickert that have a neutral or semi-attractive fixed point with an attracting set biholomorphic to $\mathbb{C}^k$ all have the fixed point lying in the boundary of the basin. It is natural to ask whether the attracting set of such a fixed point can ever contain an open neighborhood of the fixed point. The following result shows that this cannot happen.

Theorem 2. *Let $f : \mathbb{C}^k \to \mathbb{C}^k$ be a holomorphic map such that $f(0) = 0$, and let Ω be the attracting set. If Ω contains a neighborhood of the origin then 0 is an attracting fixed point.*

Proof. Suppose for the purpose of a contradiction that the closed ball $\overline{B}_r$ is in the attracting set for some $r > 0$, and that 0 is not an attracting fixed point. It follows that no iterate of f has 0 as an attracting fixed point.

Our first claim is that the set $\{f^n(\overline{B}_r)\}_{n\in\mathbb{N}}$ is unbounded. If not then the set of iterates $\{f^n\}$ would be a normal family on $\overline{B}_r$, and we may pass to a convergent subsequence. But since $f^n(x)\to\ 0$ for all $x\in\overline{B}_r$ we get that $f^N(\overline{B}_r)\subset\subset\overline{B}_r$ for some $N\in\mathbb{N}$, and by the contraction principle (see page 219 of [10]) this contradicts the fact that 0 is not an attracting fixed point for f^N.

For $m\in\mathbb{N}$ define the bounded set $K_m:=\cup_{i=1}^m f^i(\overline{B}_r)$. It follows from the above claim that there must be a point $x\in\overline{B}_r$ such that $f^n(x)\in\mathbb{C}^k\setminus K_m$ for some $n\in\mathbb{N}$, so it follows that in the sequence $\{f(x),f^2(x),...,f^n(x)\}$ there have to be at least m successive $f^i(x)$'s such that $f^i(x)\notin B_r$. So each set $C_m=\{x\in\overline{B}_r\mid f^j(x)\notin B_r, j=1,...,m\}$ is non-empty. We have that $C_1=f^{-1}(\mathbb{C}^k\setminus B_r)\cap\overline{B}_r$, and then $C_m=f^{-m}(\mathbb{C}^k\setminus B_r)\cap C_{m-1}$ for $m=2,3,...$, so we have a decreasing sequence of compact sets. Therefore there is a point $x\in\cap_{i=1}^\infty C_i$, and it follows that $f^j(x)$ does not converge to the origin, which is a contradiction. $\square$

Example 1. *Theorem 2 does not hold for holomorphic self maps of complex manifolds in general. The mapping* $f(z)=\frac{z}{1+z}$ *is an automorphism of the Riemann sphere and the orbit* $f^m(z)$ *is given by* $f^m(z)=\frac{z}{1+mz}$*. So we see that the origin is an attracting fixed point for* f*, and the attracting set is in fact equal to the entire Riemann sphere. But since* $f^m(\frac{-1}{m})=\infty$ *we see that the attraction is not uniform in any neighborhood of the origin.*

Example 2. *Theorem 2 obviously does not hold for diffeomorphisms, consider for example (for* $x\in\mathbb{R}^k$*)*

$$\begin{aligned} x &\mapsto (1-\|x\|^{\frac{1}{\|x\|}})x, \text{ for } x\neq 0,\\ 0 &\mapsto 0. \end{aligned}$$

Here the basin is the unit ball, and the attraction is uniform on compact subsets. This raises the following question: If f *is a homeomorphism of* $\mathbb{R}^n$*, and suppose that* f *has a fixed point such that the attracting set contains a neighborhood of the fixed point. Is the attraction necessarily uniform on compact subsets? The answer to this question is also negative when* $n\geq 2$*.*

For $x\in\mathbb{R}$ *let*

$$\psi(x)=\frac{x(4\pi-x)}{2\pi}.$$

Notice that $\psi(0)=0,\psi(2\pi)=2\pi$ *and* $\psi^n(x)\to 2\pi$ *for any* $x\in(0,2\pi)$ *as* $n\to\infty$*.*

Now let $f:\mathbb{C}\to\mathbb{C}$ *be defined as follows:*

For $r\geq 1$ *and* $\theta\in[0,2\pi)$*, we define*

$$f(re^{i\theta})=\frac{r+1}{2}e^{i\psi(\theta)}.$$

To define $f(z)$ *for* z *inside the unit ball, note that any such* z *lies on a unique circle through* 1 *that is tangent to the unit circle. Let* f *fix those circles, so that the angle (with respect to the center of such a circle) of* $f(z)$ *becomes* $\psi(\theta)$*, where* θ *is the angle of* z*.*

Note that f *is continuous, has a fixed point at* 1 *and* $f^n(z)$ *converges to* 1 *for any* $z\in\mathbb{C}$*. Yet the convergence is not uniform. With a little care the same construction works for a diffeomorphism.*

3. Dense Stable Manifolds

Before we prove Theorem 1, we first show that an attracting set of a volume preserving automorphism cannot have interior points. We have already noted that generally the attracting set of a semi-attracting or neutral fixed point can be biholomorphic to $\mathbb{C}^k$, so can in particular have interior. However, we easily see that this cannot be the case when dealing with volume preserving automorphisms.

Proposition 1. *Let $f \in \mathrm{Aut}_1(\mathbb{C}^k)$ have a fixed point at the origin, and let Ω be the attracting set. Then Ω has empty interior.*

Proof. Let $\epsilon > 0$ and define U as the set of those points whose forward orbit lies entirely in B_ϵ. Then U is forward invariant under f, and for every $n \in \mathbb{N}$ we have that $f^n(U) \subset B_\epsilon$ and $\mathrm{Vol}(f^n(U)) = \mathrm{Vol}(U)$. It follows that for every $n \in \mathbb{N}$ we have that the set $\{z \notin B_\epsilon, f^n(z) \in U\}$ has no volume. So the countable union

$$\{z \notin B_\epsilon \mid \exists n \in \mathbb{N} : f^n(z) \in U\}$$

has empty interior. But the orbit of any point in Ω must eventually land in U. Hence the set $\Omega \setminus B_\epsilon$ has empty interior. Since this holds for any $\epsilon > 0$ the proof is complete. $\square$

For an automorphism f with a hyperbolic fixed point p denote the *local stable manifold* by

$$\Sigma^f_\epsilon(p) = \{z \in \mathbb{C}^k \mid \|f^n(z) - p\| < \epsilon \ \forall n \in \mathbb{N}\}.$$

For small enough ϵ we have that $\Sigma^f_\epsilon(p)$ is a graph over the attracting direction of $\mathrm{d}f(p)$, and $\{f^n(z)\}$ converges to p if and only if $f^n(z) \in \Sigma^f_\epsilon(p)$ for some $n \in \mathbb{N}$ (see for instance Chapter 6.2 in Katok-Hasselblat [9]). In other words, if we denote the attracting set or *stable manifold* by $\Sigma^f(p)$ then

$$\Sigma^f(p) = \bigcup_{n\in\mathbb{N}} f^{-n}\Sigma^f_\epsilon(p).$$

As noted before, it follows from the appendix of [13] that $\Sigma^f(p)$ is biholomorphic to $\mathbb{C}^m$, where m is the number of attracting directions.

To prove Theorem 1 we need a stability condition for stable manifolds. Let us fix an $f \in \mathrm{Aut}_1(\mathbb{C}^k)$ with a fixed point at the origin, and assume that f is of the form

$$f(z) = (\lambda_1 z_1 + \alpha_1(z), \ldots, \lambda_m z_m + \alpha_m(z), \mu_1 z_{m+1} + \alpha_{m+1}(z), \ldots, \mu_{k-m} z_k + \alpha_k(z)),$$

where $|\lambda_i| < 1$, $|\mu_i| > 1$, and the α_i's are functions of degree at least two. For $\delta > 0$ we let $\triangle^m_\delta$ denote the polydisk $\triangle^m_\delta = \{(z_1, \cdots, z_k) \in \mathbb{C}^k \mid z_i = 0 \text{ for } i > m, |z_i| < \delta \text{ for } i \leq m\}$. As stated above: If ϵ is small enough then for all $\delta < \epsilon$ we have that $\Sigma^f_\epsilon(0)$ is (locally) a graph Γ^f_δ over $\triangle^m_\delta$. We need the following proposition:

Lemma 1. *Let $\{f_j\} \subset \mathrm{Aut}_1(\mathbb{C}^k)$ such that $\|f - f_j\|_{\overline{\triangle^k_\epsilon}} \to 0$ as $j \to \infty$. Then there exists a fixed $\delta < \epsilon$ such that for all j large enough:*

(a) f_j has a unique hyperbolic fixed point p_j in $\triangle^k_\epsilon$, and $p_j \to 0$ as $j \to \infty$,
(b) $\Sigma^{f_j}_\epsilon(p_j)$ is (locally) a graph $\Gamma^{f_j}_\delta$ over $\triangle^m_\delta$,
(c) $\mathrm{d}(\Gamma^{f_j}_\delta, \Gamma^f_\delta) \to 0$ as $j \to \infty$, where $\mathrm{d}(\cdot,\cdot)$ denotes the Hausdorff distance.

Sketch of the proof. (a) is well known, and for (b) and (c) we may well assume that $p_j = o$ for high enough j.

In a small enough polydisc Δ_δ, the map f is strictly expanding in the repelling directions and strictly contracting in the attracting direction. This gives that $\Sigma^f_\delta(0)$ is a graph over the attracting direction. For f_j close enough to f we have that f_j is still strictly expanding and contracting in this same polydisc and we get (b).

For $\gamma > 0$ arbitrarily small, let $\mathcal{N}_\gamma$ be the γ neighborhood of $\Sigma^f_\delta(0)$, and let $K = \Delta_\delta - \mathcal{N}_\gamma$. Then there is an $n \in \mathbb{N}$ such that for every $z \in K$ there is an $j \in \{1, \cdots, n\}$ such that $f^j(z) \notin \Delta_\delta$. Hence the same is true for f_j close enough, so we have that $\Sigma^{f_j}_\delta(0) \subset \mathcal{N}_\gamma$. But since $\Sigma^{f_j}_\delta(0)$ and $\Sigma^f_\delta(0)$ are both graphs over the stable direction, we must have that $\mathrm{d}(\Sigma^f_\delta(0), \Sigma^{f_j}_\delta(0)) < \gamma$. $\square$

For each $n \in \mathbb{N}$ we now let $\Gamma^{f_j}_\delta(n)$ denote the set $f_j^{-n}(\Gamma^{f_j}_\delta)$, such that $\Sigma^{f_j}(p_j) = \bigcup_{n\in\mathbb{N}} \Gamma^{f_j}_\delta(n)$. The following is then an immediate consequence of the above proposition:

Corollary 1. *Let U be any neighborhood of $\overline{\triangle}^k_\epsilon \cup \Gamma^f_\delta(n)$ for some fixed $n \in \mathbb{N}$, and let $\{f_j\} \subset \mathrm{Aut}_1(\mathbb{C}^k)$ such that $\|f - f_j\|_{\overline{U}} \to 0$ as $j \to \infty$. Then $\mathrm{d}(\Gamma^{f_j}_\delta(n), \Gamma^f_\delta(n)) \to 0$ as $j \to \infty$.*

Proposition 2. *Let $f \in \mathrm{Aut}_1(\mathbb{C}^k)$ as above have a hyperbolic fixed point at the origin, let $\delta, \rho > 0$, let $q \in \mathbb{C}^k$ and let K be a compact subset of $\mathbb{C}^k$. Then there exists a $g \in \mathrm{Aut}_1(\mathbb{C}^k)$ such that the following hold:*

(a) g has a unique hyperbolic fixed point p close to the origin,
(b) $\|g - f\|_K < \delta$,
(c) There is a point $q' \in \Sigma^g(p)$ such that $\|q' - q\| < \rho$.

Proof. We assume that (c) is not already satisfied by $\Sigma^f(0)$, and we assume that K is a closed ball. By Theorem 3.1 in [3] there is a $\tilde{g} \in \mathrm{Aut}_1(\mathbb{C}^k)$ with $\|\tilde{g} - f\|_K < \frac{\delta}{2}$ and such that the unbounded orbits of $\tilde{g}$ are dense in $\mathbb{C}^k$. Choose $\tilde{q} \in \mathbb{C}^k$ with $\|\tilde{q} - q\| < \frac{\rho}{2}$ such that $\{\tilde{g}^n(\tilde{q})\}_{n\in\mathbb{N}}$ is unbounded. By Lemma 1 we know that if $\tilde{g}$ is a good enough approximation of f then g has a hyperbolic fixed point $\tilde{p}$ near 0. We have that $\Sigma^{\tilde{g}}(\tilde{p})$ is unbounded since $\Sigma^{\tilde{g}}(\tilde{p})$ is biholomorphic to $\mathbb{C}^m$ (This follows from the appendix of [13]).

So we may choose a point $x \in \Sigma^{\tilde{g}}(\tilde{p})$ such that $x \in \mathbb{C}^k \setminus K$ and such that $\tilde{g}^n(x) \in K$ for all $n \geq 1$. Let M be an integer such that $\tilde{g}^M(x) \in \Gamma^{\tilde{g}}_\delta$, and let N be the smallest integer such that $\tilde{g}^N(\tilde{q}) \subset \mathbb{C}^k \setminus K$. For some $r > 0$ we have that the set $K \cup B_r(x) \cup B_r(\tilde{g}^N(\tilde{q}))$ is polynomially convex, and we let $\phi \in \mathrm{Aut}_1(\mathbb{C}^k)$ such that $\phi(B_r(\tilde{g}^N(\tilde{q})) = B_r(x)$. Let $\mathcal{N}$ be a small enough neighborhood of $\tilde{q}$ such that $\tilde{g}^N(\mathcal{N}) \subset\subset B_r(\tilde{g}^N(\tilde{q}))$, and let $\mathcal{V}$ be a small enough neighborhood of $\tilde{g}^M(x)$ such that $\mathcal{V} \subset\subset \tilde{g}^M \circ \phi \circ \tilde{g}^N(\mathcal{N})$.

By [5] and [6] there is exists a sequence of automorphisms $\phi_j \in \mathrm{Aut}_1(\mathbb{C}^k)$ such that $\phi_j \to \phi$ on $B_r(\tilde{g}^N(x))$ and such that $\phi_j \to \mathrm{Id}$ on K. Approximating by a *volume preserving* automorphism is possible because of the vanishing of the following cohomology group [6]

$$H^{k-1}(K \cup \overline{B_r(\tilde{g}^N(\tilde{q}))}, \mathbb{C}) = 0.$$

Let Φ_j denote $\tilde{g} \circ \phi_j$. If j is large enough we have that $\mathcal{V} \subset\subset \Phi_j^{M+N+1}(\mathcal{N})$, and by Lemma 1 we have that $\Gamma_\delta^{\Phi_j} \cap \mathcal{V} \neq \emptyset$ if j is large. So the global stable manifold of Φ_j intersects $\mathcal{N}$, and the result follows by letting $g = \Phi_j$. $\square$

Corollary 2. *Let $q_1, ..., q_m$ be points in $\mathbb{C}^k$ and let $\epsilon > 0$. Then the set of automorphisms $f \in \mathrm{Aut}_1(\mathbb{C}^k)$ having a stable manifold $\Sigma^f(p)$ with points $p_1, ..., p_m \in \Sigma_p^f$ such that $\|p_i - q_i\| < \epsilon$ is dense and open.*

Proof. Let $f \in \mathrm{Aut}_1(\mathbb{C}^k)$. Let $N \in \mathbb{N}$ be arbitrary, $\rho > 0$, and choose $p \notin (B_N \cup f(B_N))$. Since $H^{k-1}(f(B_N), \mathbb{C}) = 0$ there is a sequence of automorphisms $\{g_j\} \in \mathrm{Aut}_1(\mathbb{C}^k)$ with $g_j(f(p)) = p$ and such that $\|g_j - \mathrm{Id}\|_{f(B_N)} \to 0$ [5]. By composing with a linear map arbitrarily close to the identity if necessary, we may assume that each $g_j \circ f$ has a hyperbolic fixed point at p. If j is large we have that $\|g_j \circ f - f\|_{B_N} < \rho$, and it follows that the set of volume preserving automorphisms having a hyperbolic fixed point is dense. By Proposition 1 it is also open.

Note that by Corollary 1 the set of volume preserving automorphisms having a stable manifold with a point p_i that is ϵ-close to some point q_i is open. Therefore it is enough to consider the point q_1. Let h denote $g_j \circ f$ for a some large j. By Proposition 2 there exists for any $\rho > 0$ a $\tilde{h} \in \mathrm{Aut}_1(\mathbb{C}^k)$ such that $\|\tilde{h} - h\|_{B_N} < \rho$ and such that $\|p_1 - q_1\| < \rho$ for some $p_1 \in \Sigma_p^{\tilde{h}}$, where p is a hyperbolic fixed point for $\tilde{h}$. The result follows.

$\square$

In the following proof, note that $\mathrm{Aut}_1(\mathbb{C}^k)$ is a *Baire Space*, meaning that a countable intersection of open and dense sets is again dense.

Proof of Theorem 1. Let $\{q_i\}_{i\in\mathbb{N}}$ be a dense set of points in $\mathbb{C}^k$ and let $\epsilon_j \searrow 0$. For each $j \in \mathbb{N}$ let $\mathcal{V}_j$ denote the set of automorphisms $f \in \mathrm{Aut}_1(\mathbb{C}^k)$ such that f has a stable manifold Σ_p^f with points $p_1, ..., p_j \in \Sigma_p^f$ and $\|p_i - q_i\| < \epsilon_j$. According to Corollary 2 each $\mathcal{V}_j$ is open and dense. Since $\mathcal{V} := \cap_{j\in\mathbb{N}} \mathcal{V}_j$ is dense the result follows. $\square$

4. Automorphisms of $\mathbb{C}^2$ tangent to the identity

We will now show that dense attracting sets also occur for volume preserving automorphisms that are tangent to the identity. We will restrict ourselves to automorphisms of $\mathbb{C}^2$, and we will see that a statement analogous to Theorem 1 holds for volume preserving automorphisms tangent to the identity. Since the proof is almost identical to the proof of Theorem 1 we will not show it in great detail. The main difficulty is to prove that the attracting set is (locally) stable under small perturbations.

We let $\mathrm{Aut}_1^1(\mathbb{C}^2, 0)$ be the set of volume preserving automorphisms of $\mathbb{C}^2$ that are tangent to the identity. We equip $\mathrm{Aut}_1^1(\mathbb{C}^2, 0)$ with the compact open topology as before. For $f \in \mathrm{Aut}_1^1(\mathbb{C}^2, 0)$ we write $f(z) = z + P_2(z) + \dots$ where $P_2(z)$ is homogeneous of degree 2. Recall from [8] that $v \in \mathbb{C}^2$ is called a characteristic direction if $P_2(v) = \lambda v$ for some $\lambda \in \mathbb{C}$. If $\lambda \neq 0$ then v is a *non-degenerate* characteristic direction.

We first claim that every automorphism tangent to the identity must have a characteristic direction. If $P_2(z) \equiv 0$ then it is clear, so without loss of generality

we assume that $P_2 = (p, q)$ with $p(x, y) \neq 0$ for some (x, y). We blow up $y = ux$ and we get that P_2 gives the rational function

$$u \mapsto \frac{q(x, ux)}{p(x, ux)} = \frac{q(1, u)}{p(1, u)}. \tag{1}$$

Note that this function must have a fixed point, which is a characteristic direction.

The subset of $\mathrm{Aut}_1^1(\mathbb{C}^2, 0)$ consisting of those automorphisms that have a non-degenerate characteristic direction is open and dense. More precisely, we have

Lemma 2. *Let $f = \mathrm{Id} + P_2 + h.o.t. \in \mathrm{Aut}_1^1(\mathbb{C}^2, 0)$ with $P_2(v) = \lambda v$. Then*
(a) If $\lambda \neq 0$ and $\tilde{f}$ is close enough to f then $\tilde{f}$ has a has a non-degenerate characteristic direction arbitrarily close to v.
(b) If $\lambda = 0$ then there exist mappings in $\mathrm{Aut}_1^1(\mathbb{C}^2, 0)$ arbitrarily close to f that have v as a non-degenerate characteristic direction.

Proof. Assertion (a) follows immediately from the fact that the characteristic directions are fixed points of the rational equation (1), whose parameters depend continuously on the mapping f.

To prove assertion (b), assume without loss of generality that $v = (1, 0)$, in other words so that $P_2(1, 0) = (0, 0)$. Let $\psi_\epsilon = \mathrm{Id} + \Psi_\epsilon + h.o.t. \in \mathrm{Aut}_1^1(\mathbb{C}^2, 0)$ with $\Psi_\epsilon(x, y) = (\epsilon x^2 m, -2\epsilon xy)$, the existence of such an automorphism follows from [1]. Then $\psi_\epsilon \circ f$ has $(1, 0)$ as non-degenerate characteristic direction. □

Now let $f \in \mathrm{Aut}_1^1(\mathbb{C}^2, 0)$ have a non-degenerate characteristic direction v. After a suitable conjugation by an affine mapping we have that $v = (1, 0)$ and that $P_2(v) = v$, i.e. $\lambda = 1$.

Write $P_2(z) = (p(z), q(z))$. Since f is volume preserving we must have that $p_x = -q_y$. Hence P_2 must be of the form

$$P_2(x, y) = (x^2 + 2bxy + cy^2, -2xy - by^2). \tag{2}$$

From here on we will assume that $b \neq 0$, the case $b = 0$ is almost identical. We can now further simplify Equation (2) by conjugating with $(x, y) \to (x, b^{-1}y)$ to get

$$P_2(x, y) = (x^2 + 2xy + cy^2, -2xy - y^2). \tag{3}$$

As in [8] we blow-up $y = ux$ and write $(x_n, u_n) = F^n(x, u)$ to get

$$x_1 = x + (1 + 2u + cu^2)x^2 + O(|x|^3) \tag{4}$$

$$u_1 = u - (3u + 3u^2 + cu^3)x + O(|x|^2) \tag{5}$$

Recall that in the hyperbolic case the stable manifold is locally a graph over the attracting direction, and that the mapping is expanding in the repelling direction. The expansion guarantees that the local stable manifold is stable under small perturbations. We will see that the situation is analogous for volume preserving automorphisms tangent to the identity: for some $\epsilon > 0$ small enough the attracting set is a graph over the set $\{x \in \mathbb{C} \mid \max(|x|, |\arg(x) - \pi|) < \epsilon\}$.

For $\epsilon > 0$ define

$$W_\epsilon = \{(x, u) \in \mathbb{C}^2 \mid \max(|x|, |\arg(x) - \pi|) < \epsilon, 2|u| < |x|\} \tag{6}$$

The following Lemma follows from Equations (4) and (5).

Lemma 3. *Let $\epsilon > 0$ small enough and let $(x,u),(\tilde{x},\tilde{u}) \in W_\epsilon$. Suppose that $2|x-\tilde{x}| < |u-\tilde{u}|$. Then $|u_1 - \tilde{u}_1| > \max(|u-\tilde{u}|, 2|x_1-\tilde{x}_1|)$.*

We also have

Lemma 4. *Let $\epsilon > 0$ be small enough and $(x,u) \in W_\epsilon$. If $(x_n,u_n) \in W_\epsilon$ for every n then $(x_n,u_n) \to (0,0)$.*

Proof. It follows from (4) that $\{x_n\}$ must converge to 0. But then $u_n \to 0$ by definition of W_ϵ. □

Proposition 3. *Let $\epsilon > 0$ be small enough and $x \in \mathbb{C}$ satisfy $|x| < \epsilon$ and $|\arg(x) - \pi| < \epsilon$. Then there is exactly one $u \in \mathbb{C}$ such that $(x_n,u_n) \in W_\epsilon$.*

Proof. To show existence, let $U_n = \{u \in \mathbb{C} \mid (x_j,u_j) \in W_\epsilon, j = 1,\cdots,n\}$. It follows from Lemma 3 that $\{U_n\}$ is a nested sequence of non-empty relatively compact sets, hence the intersection is not empty.

To prove uniqueness, suppose for the purpose of contradiction that there exist two such points, u and v. Then Lemma 3 shows inductively that $|u_n - v_n| > |u-v|$ for every n, but Lemma 4 shows that both u_n and v_n must converge to the origin, so we have a contradiction. □

So we indeed have that the attracting set is locally a graph over the attracting direction, and we have expansion is the other direction. Stability follows:

Theorem 3. *Let $f \in \mathrm{Aut}_1^1(\mathbb{C}^2,0)$ have a non-degenerate attracting direction, and let Ω be the corresponding attracting set. If $\{f_j\} \subset \mathrm{Aut}_1^1(\mathbb{C}^2,0)$ converges to f, then the corresponding attracting sets Ω_j satisfy (locally)*

$$\mathrm{d}(\Omega_j,\Omega) \to 0. \tag{7}$$

To prove Theorem 3, first assume that for j large enough the maps f_j all have a fixed point at the origin, and the same non-degenerate direction $(1,0)$. We can do this because for j large enough this can be assured by conjugating with an affine map arbitrarily close to the identity.

It follows from Lemmas 3, 4 and Proposition 3 that the proof of Lemma 1 works here as well, with expansion now in the u-coordinate instead of the y-coordinate.

We obtain the existence of dense attracting sets for volume preserving automorphisms tangent to the identity.

Theorem 4. *There is a dense G_δ-set $\mathcal{S} \subset \mathrm{Aut}_1^1(\mathbb{C}^2,0)$ such that each $f \in \mathcal{S}$ has a fixed point tangent to the identity whose attracting set is dense in $\mathbb{C}^2$.*

As the proof is very similar to the proof of Theorem 1 we will only outline the differences. The unboundedness of the attracting sets follows from Theorem 1.10 of [7], which implies that the attracting set is conformally equivalent to $\mathbb{C}$.

To complete Theorem 4 we need to confirm that a version of Theorem 3.1 from [3] holds for automorphisms tangent to the identity. Let $\mathcal{V}$ denote the set of volume preserving automorphisms tangent to the identity at the origin that have a non-degenerate characteristic direction. For each $f \in \mathcal{V}$ we define as in [3]:

$$K_f = \{z \in \mathbb{C}^2 \mid \{f^n(z)\} \text{ is bounded}\}.$$

The claim in [3] becomes: *There exists a dense G_δ set $\mathcal{V}_1$ in $\mathcal{V}$ such that for every $f \in \mathcal{V}_1$, the set K_f is an F_σ set with empty interior.* We outline the additions that have to be made to prove the claim.

For $f_0 \in \mathcal{V}$, we define U_C as the interior of the set of points whose forward orbits are contained in the ball B_C. Note that the set U_C is forward invariant under f_0. Since f_0 has a non-degenerate characteristic direction at the origin, we have a repelling piece of curve (namely, the attracting set to the origin for f^{-1}). This curve is unbounded (since it is biholomorphic to $\mathbb{C}$), so the origin cannot lie not in U_C.

When the existence of the polynomially convex set $X_q \subset U_C$ (as in [3]) is established, we may assume that $X_q \cap \{0\} = \emptyset$. This follows because X_q is the orbit of q under the action of a commutative compact Lie group acting as automorphisms on U_C. Hence we may extend the vector field ξ to be zero on a neighborhood of the origin, and one can still approximate ξ by divergence free polynomial vector fields.

When approximating the flow of ξ by the 1-parameter family ψ_t of volume preserving automorphisms, we may then assume that ψ_t is tangent to the identity for each t, by composing with the inverses of the derivatives at the origin. The claim follows just as in [3].

5. The non-autonomous case

For many results in complex dynamical systems it makes sense to ask whether the result also holds in the non-autonomous setting. Instead of studying the iterations $\{f^n\}$ of a fixed mapping, one studies the compositions $\{f_n \circ \cdots \circ f_1\}$ for a sequence of mappings $f_1, f_2, \ldots$. As this gives much more freedom, it is generally easier to construct (counter-) examples, but harder to prove that general results still hold in the autonomous setting.

For a sequence $f_1, f_2, \ldots \in \mathrm{Aut}(\mathbb{C}^k)$ that all have a fixed point at the origin one can define the attracting set as

$$\Omega^{\{f_j\}} = \{z \in \mathbb{C}^k \mid f_n \circ \cdots \circ f_1(z) \to 0\}$$

The complex structure of such basins has been studied in [2], [12], [11]. The following construction shows that in the non-autonomous setting one can have an attracting set contains a neighborhood of the origin for a sequence of automorphisms that are all tangent to the identity.

Theorem 5. *There exists a sequence $\{g_j\}_{j=1}^\infty$ of automorphisms of $\mathbb{C}^2$ with $g_j(0) = 0, \mathrm{d}g_j(0) = \mathrm{Id}$ for all $j \in \mathbb{N}$, and such that $\Omega^{\{g_j\}} = \mathbb{C}^2$.*

Instead of $\mathrm{d}g_j(0) = \mathrm{Id}$ we may in fact freely prescribe the d-jets for each of the mappings $\{g_j\}$ and for any fixed d. For any sequence of d-jets there exists a sequence of automorphisms having these d-jets and that satisfies the requirements needed in the proof [4] [15].

The same construction works in dimensions higher than 2.

Proof. The idea of the proof is simple. We construct an increasing sequence of subsets $U_1 \subset U_2 \subset \ldots$ of $\mathbb{C}^2$ whose union is $\mathbb{C}^2$, and we construct automorphisms $f_1, f_2, \ldots$ that are all tangent to the identity such that f_j maps U_j into a (smaller and smaller) neighborhood of the origin. Then we define $g_1 = f_1$ and $g_j = f_j \circ f_{j-1}^{-1}$ for $j \geq 2$, and we are done.

For $R > 0$ and some small $\epsilon > 0$ we define the following subsets of the complex plane:

- $\overline{\triangle}_R = \{|z| \leq R\}$,
- $\Theta_R^\epsilon := \{re^{i\theta} \in \mathbb{C} \mid -\epsilon < \theta < \epsilon,\ 0 \leq r \leq R\}$,
- $K_R^\epsilon := \overline{\triangle}_R \setminus \Theta_R^\epsilon - \epsilon$ (here $-\epsilon$ is a translation to the left),
- $L_R^\epsilon := \{x + iy \in \mathbb{C} \mid y = 0,\ \epsilon \leq x \leq R\}$.

For $\delta > 0$ small enough we have that the following set is a union of five disjoint compact sets in $\mathbb{C}^2$:

$$N_{R,\delta}^\epsilon := \overline{B}_\delta(0,0) \cup (K_R^\epsilon \times \overline{\triangle}_R) \cup (L_R^\epsilon \times \{0\}) \cup (\{0\} \times K_R^\epsilon) \cup (\{0\} \times L_R^\epsilon).$$

We claim that $N_{R,\delta}^\epsilon$ is polynomially convex. To see this, note first that the set $K_R^\epsilon \times \overline{\triangle}_R \cup (\overline{\triangle}_\delta \times \overline{\triangle}_R)$ is polynomially convex. Since $\overline{B}_\delta(0,0) \subset \overline{\triangle}_\delta \times \overline{\triangle}_R$ it follows by an application of the Oka-Weil theorem that $K_R^\epsilon \times \overline{\triangle}_R \cup \overline{B}_\delta(0,0)$ is polynomially convex. In adding the rest of the components, we add compact sets contained in closed submanifolds of $\mathbb{C}^2$ (in fact complex lines), so it suffices to check that the intersection of $N_{R,\delta}^\epsilon$ with these submanifolds is polynomially convex. (to see that this suffices, first use the defining function, and then apply the local maximum modulus principle). That the intersection with the complex lines is polynomially convex follows from Runge's theorem.

Pick $\rho > 0$ small enough such that the balls $\overline{B}_\delta(0,0)$ and $\overline{B}_\rho(\cdot,\cdot)$ are pairwise disjoint, and their union is polynomially convex. Then define automorphisms $\phi_1, ..., \phi_5$ of $\mathbb{C}^2$ such that the following holds:

(i) $\phi_1|_{\overline{B}_\delta(0,0)} = \mathrm{Id}$,
(ii) $\phi_2(K_R^\epsilon \times \overline{\triangle}_R) \subset B_\rho(-\epsilon, 0)$,
(iii) $\phi_3(L_R^\epsilon \times \{0\}) \subset B_\rho(\epsilon, 0)$,
(iv) $\phi_4(\{0\} \times K_R^\epsilon) \subset B_\rho(0, -\epsilon)$,
(v) $\phi_5(\{0\} \times L_R^\epsilon) \subset B_\rho(0, \epsilon)$.

Choose a small neighborhood U of $N_{R,\delta}^\epsilon$ and define a map $\psi : U \to \mathbb{C}^2$ such that $\psi|_{\overline{B}_\delta(0,0)} = \phi_1, \psi|_{K_R^\epsilon \times \overline{\triangle}_R} = \phi_2$, and so forth. By [5] the map ψ may be approximated arbitrarily well on $N_{R,\delta}^\epsilon$ by an automorphism $f = f_{R,\delta}^\epsilon$ of $\mathbb{C}^2$. By composing with a linear map close to the identity if necessary, we may assume that $f(0) = 0, \mathrm{d}f(0) = \mathrm{Id}$. Note that if the approximation of ψ is good enough then the image of $N_{R,\delta}^\epsilon$ is contained in the ball $B_{2\epsilon}(0,0)$.

To finish the proof we now choose sequences $R_j, \epsilon_j, \delta_j$ for $j = 1, 2, ...$, such that $\epsilon_j, \delta_j, \rho_j \searrow 0$ and $R_j \nearrow \infty$, and such that we can carry out the above construction for the sets $N_{R_j,\delta_j}^{\epsilon_j}$ to get a sequence of automorphisms $f_j = f_{R_j,\delta_j}^{\epsilon_j}$ as above. Now define the sequence g_j inductively by $g_1 = f_1$ and $g_j = f_j \circ f_{j-1}^{-1}$ for $j = 2, 3, ...$

We have that the union of the increasing sequence $N_{R_j,\delta_j}^{\epsilon_j}$ is all of $\mathbb{C}^2$ and we have

$$g_j g_{j-1} \cdots g_1(N_{R_j,\delta_j}^{\epsilon_j}) = f_j(N_{R_j,\delta_j}^{\epsilon_j}) \subset B_{2\epsilon_j}(0,0).$$

This completes the proof. □

Remark 1. Note that the convergence in the above theorem is *pointwise* - not uniform on compacts. Specifically we have that points arbitrarily close to the

origin go arbitrarily far out towards infinity. In light of this example one might ask whether there exist an attracting set of a non-periodic bounded orbit $p_0, p_1, \ldots$, such that the attracting set contains a neighborhood of p_0. The arguments in the proof of Theorem 2 adapts to this case however, showing that this is impossible.

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